Review Problems for "Applied Functional Analysis"

1. Let M be an inner product space, on which the norm is derived from inner products via

$$\|x\| = \sqrt{\langle x, x\rangle}.$$

Prove that for any $x, y \in M$, we have the following Parallelogram identify:

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Proof. Note that $\|x+y\|^2 = \langle x+y, x+y\rangle$ and $\|x-y\|^2 = \langle x-y, x-y\rangle$. We have

$$\begin{aligned}
\|x+y\|^2 + \|x-y\|^2 &= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \\
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle + \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle \\
&= 2\langle x, x\rangle + 2\langle y, y\rangle \\
&= 2\|x\|^2 + 2\|y\|^2.
\end{aligned}$$

Done! □

2. On $\mathbb{C}^2$, define two norms as

$$\|(x,y)\|_1 = \sqrt{|x|^2 + |y|^2} \quad \text{and} \quad \|(x,y)\|_2 = |x| + |y|.$$

Prove that these two norms are equivalent. In other words, prove that there exist $0 < C_1 < C_2$, such that for all $(x,y) \in \mathbb{C}^2$,

$$C_1 \|(x,y)\|_2 \le \|(x,y)\|_1 \le C_2 \|(x,y)\|_2 .$$

Proof. As $\sqrt{|x|^2 + |y|^2} \le |x| + |y|$, we have

$$\|(x,y)\|_1 \le \|(x,y)\|_2 , \ \forall\, (x,y) \in \mathbb{C}^2.$$

Note that $\sqrt{|x|^2 + |y|^2} \ge |x|$ and $\sqrt{|x|^2 + |y|^2} \ge |x|$, we get

$$\sqrt{|x|^2 + |y|^2} \ge (|x| + |y|)/2, \ \forall\, (x,y) \in \mathbb{C}^2.$$

In other words, we have

$$\frac{1}{2} \|(x,y)\|_2 \le \|(x,y)\|_1 , \ \forall\, (x,y) \in \mathbb{C}^2,$$

which finishes the proof.

□

3. Prove that $l^1 = \{(x_1, x_2, \cdots)\colon \sum_{i=1}^{\infty} |x_i| < \infty\}$ is not an inner product space. In other words, show that the norm on l^1, which is defined as

$$\|x\| = \sum_{i=1}^{\infty} |x_i|,$$

is not a norm derived from certain inner product.

Proof. In order to show that this norm is not a norm derived from certain inner product, we just need to check that the parallelogram fails for certain $x, y \in l^1$.

Let $x = (1, 0, 0, 0, \cdots)$ and let $y = (0, 1, 0, 0, \cdots)$. Then $x + y = (1, 1, 0, 0, \cdots)$ and $x - y = (1, -1, 0, 0, \cdots)$. Easy to check that

$$\|x\| = 1, \|y\| = 1, \|x + y\| = 2, \text{ and } \|x - y\| = 2.$$

But we don't have

$$2^2 + 2^2 = 2(1^2 + 1^2),$$

which finishes the proof. □

4. On $\mathbb{C}^2$, for any $x = (x_1, x_2)$ and $y = (y_1, y_2)$, define $\rho(x, y) = |x_1 - y_1|^2 + |x_2 - y_2|^2$. Prove that ρ is not a metric on $\mathbb{C}^2$ (thus it cannot be a norm on $\mathbb{C}^2$).

Proof. We will check against the triangle inequality, showing that it does not always hold true, which will then suffice for the proof.

Let $x = (x_1, x_2) = (0, 0)$, $y = (y_1, y_2) = (1, 1)$ and let $z = (z_1, z_2) = (2, 2)$. We can check that

$$\rho(x, y) = (1 - 0)^2 + (1 - 0)^2 = 2$$

$$\rho(x, z) = (2 - 0)^2 + (2 - 0)^2 = 8$$

$$\rho(y, z) = (2 - 1)^2 + (2 - 1)^2 = 2.$$

As

$$\rho(x, z) > \rho(x, y) + \rho(y, z),$$

the triangle inequality does not hold for this ρ. Thus ρ is not a metric on $\mathbb{C}^2$. □

5. Define

$$C^b(\mathbb{R}) = \{f \in C(\mathbb{R})\colon \sup_{x \in \mathbb{R}} |f(x)| < \infty\}.$$

For any $f \in \mathbb{C}^(\mathbb{R})$, define its norm to be

$$\|f\| = \sup_{x \in \mathbb{R}} f(x).$$

Prove that $(C^b(\mathbb{R}), \|\|)$ is a normed space, but NOT an inner product space.

Proof. For any $f, g \in C^b(\mathbb{R})$ and $\lambda \in \mathbb{R}$, easy to check that

$$\|f\| = 0 \text{ if and only if } f = 0$$

and

$$\|\lambda f\| = |\lambda| \cdot \|f\|.$$

It remains to show that $\|f+g\| \leq \|f\| + \|g\|$. In fact, for all $x \in \mathbb{R}$

$$|(f+g)(x)| = |f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \sup_{x\in\mathbb{R}} |f(x)| + \sup_{x\in\mathbb{R}} |g(x)|,$$

which then implies that

$$\|f+g\| = \sup_{x\in\mathbb{R}} |(f+g)(x)| \leq \|f\| + \|g\|.$$

Now, we will show that this norm can not be derived from any inner product. It suffices to show that the parallelogram identity does not hold.

Let

$$f(x) = \begin{cases} 1 & x \in [1, \infty) \\ x & x \in (-1, 1) \\ -1 & x \in (-\infty, -1] \end{cases}.$$

One can check that

$$\|f\| = 1, \|1\| = 1, \|f+1\| = 2, \text{ and } \|f-1\| = 2,$$

and we don't have

$$\|f+1\|^2 + \|f-1\|^2 = 2(\|f\|^2 + \|1\|^2).$$

□

6. On $\mathbb{C}^2$, define the norm of $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2$ to be $\|(x,y)\| = \sqrt{x^2+y^2}$. For a 2×2 matrix, regard it as an linear operator on $\mathbb{C}^2$ defined as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}.$$

Prove that this linear operator $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a bounded linear operator, and the operator norm satisfies

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\| \leq |a| + |b| + |c| + |d|.$$

Proof. Just need to show the following: for any $x, y \in \mathbb{C}$ with $|x|^2 + |y|^2 \leq 1$,

$$|ax+by|^2 + |cx+dy|^2 \leq (|a| + |b| + |c| + |d|)^2.$$

Note that $2|x||y| \leq |x|^2 + |y|^2 \leq 1$, we have

$$\begin{aligned}|ax+by|^2 + |cx+dy|^2 &= |a|^2 + 2|x||y||a||b| + |b|^2 + |c|^2 + 2|x||y||c||d| + |d|^2 \\ &\leq |a|^2 + |a||b| + |b|^2 + |c|^2 + |c||d| + |d|^2 \\ &\leq |a|^2 + |b|^2 + |c|^2 + |d|^2 + 2|a||b| + 2|c||d| \\ &\qquad + 2|a||c| + 2|a||d| + 2|b||c| + 2|b||d| \\ &= (|a| + |b| + |c| + |d|)^2.\end{aligned}$$

□

7. This is about finding a basis for finite dimensional Hilbert spaces, and about the properties of such basis.

On $\mathbb{R}^2$, define the inner product as

$$\langle x, y\rangle = x_1y_1 + x_2y_2,\ \forall\, x, y \in \mathbb{R}^2,\ \text{with } x = (x_1, x_2) \text{ and } y = (y_1, y_2).$$

It is easy to check that $\mathbb{R}^2$ is a Hilbert space under this inner product. Assume we have $e_1 = (\frac{3}{5}, \frac{4}{5})$. Easy to check that $\|e_1\| = 1$.

a) Find $e_2 \in \mathbb{R}^2$ such that

$$\|e_2\| = 1 \quad \text{and} \quad \langle e_1, e_2\rangle = 0.$$

b) For the e_1, e_2 we have so far, prove that for any $x \in \mathbb{R}^2$,

$$\|x\|^2 = \langle x, x\rangle = \langle x, e_1\rangle^2 + \langle x, e_2\rangle^2.$$

Solution:

a) First, we try to find $x \in \mathbb{R}^2$ that is not in the linear span of e_1.

Choose $x = (1, 0)$. It is clear that $x \notin \{\lambda e_1 \colon \lambda \in \mathbb{R}\}$. Let

$$\begin{aligned}y &= x - \langle x, e_1\rangle e_1 \\ &= (1,0) - \left\langle (1,0), (\frac{3}{5}, \frac{4}{5})\right\rangle \cdot (\frac{3}{5}, \frac{4}{5}) \\ &= (1,0) - (\frac{9}{25}, \frac{12}{25}) \\ &= (\frac{16}{25}, -\frac{12}{25}).\end{aligned}$$

To get e_2, we just need to normalize y. That is, set

$$e_2 = \frac{y}{\|y\|} = \frac{(\frac{16}{25}, -\frac{12}{25})}{\sqrt{(\frac{16}{25})^2 + (-\frac{12}{25})^2}} = \frac{(\frac{16}{25}, -\frac{12}{25})}{\frac{4}{5}} = (\frac{4}{5}, -\frac{3}{5}).$$

b) The proof is just straightforward verifications.